INSTRUCTIONS:

(i) There are 12 questions. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.
Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

Question 1.
Let $V$ be a finite dimensional vector space over a field $k$. Let $W$ be a subspace.
Suppose that $T : V \rightarrow V$ is a linear transformation for which $TW \subseteq W$. Prove that $T$ induces linear transformations $W \rightarrow W$ and $V/W \rightarrow V/W$ (which we also denote by $T$). Prove that
\[ \text{trace}(T|_W) + \text{trace}(T|_{V/W}) = \text{trace}(T|_V). \]

Question 2.
Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional real inner product space with associated norm $\|x\| := \langle x, x \rangle^{1/2}$. Let $P : V \rightarrow V$ be a projection, i.e. there exist subspaces $A,B \subset V$ for which $A \oplus B = V$ and $P(a \oplus b) = a$ for every $a \in A, b \in B$.
Suppose that $P$ has operator norm 1, i.e. $1 = \sup_{x \in V : \|x\| \leq 1} \|Px\|$. Prove that $A \perp B$, i.e. $\langle a, b \rangle = 0$ for all $a \in A, b \in B$.

Question 3.
For an $n \times n$ complex matrix $T$, let $T^*$ denote the “conjugate transpose” matrix, i.e. $T^*_{ij} = \overline{T_{ji}}$. Let $A$ and $B$ be $n \times n$ complex matrices. Prove that
\[ |\text{trace}(A^*B)| \leq \text{trace}(A^*A)^{1/2} \cdot \text{trace}(B^*B)^{1/2}. \]

Question 4.
Let $V$ be a finite-dimensional, vector space over $\mathbb{C}$. Let $T : V \rightarrow V$ be a linear transformation.
(a) Suppose $p(x) \in \mathbb{C}[x]$. Let $T : V \rightarrow V$ be a linear transformation satisfying $p(T) = 0$. Suppose $p = ab$ for two polynomials $a, b$ satisfying $\gcd(a, b) = 1$. Prove that
\[ V = \ker(a(T)) \oplus \ker(b(T)). \]
(b) Let $T : V \rightarrow V$ be a linear transformation. For $\alpha \in \mathbb{C}$, define
\[ V_\alpha := \{ v \in V : (T - \alpha \cdot 1)^N v = 0 \} \text{ for some positive integer } N. \]
Prove that there is a direct sum decomposition
\[ V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha. \]
Single variable real analysis

Solve any three out of the four questions 5,6,7,8.

Question 5.
   a) Give an example of a sequence of continuous functions $f_n(x) : [0, 1] \rightarrow \mathbb{R}$ which converges pointwise on $[0, 1]$ to a continuous function $f(x)$ but not uniformly to $f$.
   b) Give an example of a sequence of integrable, continuous functions $f_n(x) : [0, \infty) \rightarrow \mathbb{R}$ which converges pointwise on $[0, \infty)$ to $f(x)$ but for which
   \[ \int_0^\infty f_n(x) \, dx \text{ does not converge to } \int_0^\infty f(x) \, dx. \]
   c) Give an example of a sequence of $C^1$ functions $f_n(x) : [0, 1] \rightarrow \mathbb{R}$ which converges uniformly to $f(x)$ on $[0, 1]$ but for which $f'_n(x)$ does not converge pointwise to $f'(x)$.

Question 6.
   Consider any sequence of real numbers $\{a_n\}$. Let $S$ be the set of all subsequential limits of $\{a_n\}$.
   a) Prove that the set $S$ is a closed subset of $\mathbb{R}$.
   b) Give an example of a sequence $\{a_n\}$ for which $S = \{0, -1, 1\}$.
   c) Give an example of a sequence $\{a_n\}$ for which $S = \mathbb{R}$.

Question 7.
   a) Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2 < \infty$. Show that $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$.
   b) Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Show that $\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n} = \infty$.

Question 8.
   Prove that a continuous function $f$ from a compact metric space $K$ to the real numbers must attain its maximum.
Solve any three out of the four questions 9, 10, 11, 12.

**Question 9.** Find the general solution $y(t) = (y_1(t), y_2(t))$ of the differential equation
\[
\begin{pmatrix}
y_1'(t) \\
y_2'(t)
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}.
\]

**Question 10.** Find the largest possible value of $y(1)$ where the function $y(x)$ satisfies $y(0) = 0$ and the differential inequality
\[y'(x) \leq 2y(x) + 4x,\]
for all points $x \in [0, 1]$.

**Question 11.** Let $h > 0$. Consider the following “cone-like” object in $\mathbb{R}^3$. The “conical top” is some region $D$ in the plane $z = h$ which lies inside a closed smooth curve $C$. The “conical side” consists of line segments from the origin $(0, 0, 0)$ to points on $C$. (Draw a picture!). Show that the volume inside this “cone” is simply $Ah/3$ where $A$ is the area of the region $D$ in the plane $z = h$. Hint: use the Divergence Theorem with a well-chosen vector field.

**Question 12.** Consider the function
\[f(x, y) = (xy)^{1/3}.
\]
At the point $(0, 0)$, show that both partial derivatives exist but $f$ is not differentiable.